

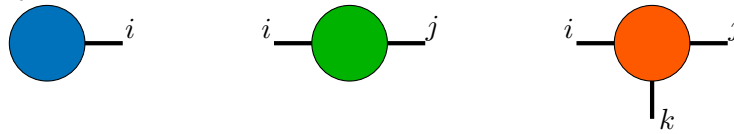
Exercises for Lattice Field Theory – Hamiltonian and Lagrangian Methods (physics7520) Sose 2026

L. Funcke, S. Singh, A. Bulgarelli and N. Dichter

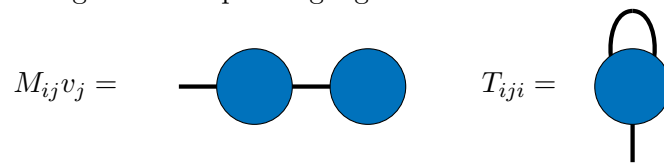
1. Tensor contractions (10 pts.)

The aim of this exercise is to introduce you to the graphical notation typically used in tensor networks. We start by recalling some basic facts. For us a tensor is an object with indices, for example a scalar ϕ is an object with 0 indices, a vector v_i has one index, taking values within the discrete set $i \in \{1, 2, \dots, D\}$, a matrix M_{ij} has two indices and so on; D is called “bond dimension”. We will always assume the Einstein sum convention $\sum_j M_{ij} v_j \equiv M_{ij} v_j$, and the components of the tensors will be in general complex numbers.

In diagrammatic notation a tensor is a geometric shape with attached a given number of legs. Each leg corresponds to an index: here are shown, respectively, a vector v_i , a matrix M_{ij} and a rank-3 tensor T_{ijk}



Indices will be omitted in the diagrams from now on. Sum over repeated indices is represented by contracting the corresponding legs



- a) (1 pts.) Draw the diagrams corresponding to matrix-matrix multiplication, the trace of a matrix, and the contraction $T_{ijk} T_{kab}$.
- b) (1 pts.) Consider a quantum state in a tensor product basis

$$|\psi\rangle = \sum_{i_1, \dots, i_N} \Psi_{i_1, \dots, i_N} |i_1, \dots, i_N\rangle,$$

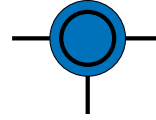
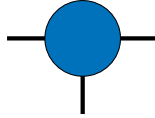
where $\Psi_{i_1, \dots, i_N}$ is the wavefunction. This for example represents the state of a spin chain with N sites, where $\{|i_a\rangle\}_{a=1, \dots, N}$ are the basis states at each site. Suppose we have d states per site, namely $i \in \{1, \dots, d\}$. For $N = 5$, represent the wavefunction as a tensor. How many independent components does it have for generic N ?

c) (1 pts.) Consider a Matrix Product State (MPS)

$$|\psi\rangle = \sum_{i_1, \dots, i_N} A_{\alpha_1}^{i_1} A_{\alpha_1 \alpha_2}^{i_2} A_{\alpha_2 \alpha_3}^{i_3} \dots A_{\alpha_{N-2} \alpha_{N-1}}^{i_{N-1}} A_{\alpha_{N-1}}^{i_N} |i_1, \dots, i_N\rangle. \quad (1)$$

Setting $N = 5$, represent the wavefunction as a contraction of rank-3 tensors, plus two rank-2 tensors at the borders. Suppose $i \in \{1, \dots, d\}$ (we will also say that the *physical bond dimension* is d) and $\alpha \in \{1, \dots, D\}$ (the *virtual bond dimension* is D). How many independent components does the MPS have for generic N ?

d) (2 pts.) Given a tensor T_{ijk}



we will denote its complex conjugate T_{ijk}^* as

Draw the diagram corresponding to the contraction $\langle\psi|\psi\rangle$, where $|\psi\rangle$ is a MPS and $N = 5$.

e) (2 pts.) Draw the diagram corresponding to the contraction $\langle\psi|\mathcal{O}_3|\psi\rangle$, where $\mathcal{O}_3$ is a matrix operator acting on the third site of the chain, and $N = 5$. Then draw the contraction $\langle\psi|\mathcal{O}_2\mathcal{O}_4|\psi\rangle$, corresponding to a correlator.

f) (3 pts.) Draw the diagram corresponding to the reduced density matrix

$$\rho_{1,2} = \text{Tr}_{3,4,5} \rho = \sum_{i_3, i_4, i_5} \langle i_3, i_4, i_5 | \psi \rangle \langle \psi | i_3, i_4, i_5 \rangle.$$

2. Efficient contraction schemes (10 pts.)

Contracting tensor networks can be costly. Here we will show that the cost of contracting a MPS grows only moderately with the bond dimension.

- (1 pts.) Consider a matrix-vector multiplication $M_{ij}v_j$. The contracted bond j has dimension D , while i has dimension d . Show that the number of operations needed for this contraction scales as $\mathcal{O}(Dd)$.
- (2 pts.) Consider two tensors, the first one with p legs, the second one with q legs, and assume $p \geq q$. Each leg has bond dimension D . What is the cost of contracting $r \leq q$ legs between the two tensors in terms of D , p , q and r ?
- (4 pts.) Show that the cost for contracting $\langle\psi|\psi\rangle$, where $|\psi\rangle$ is a MPS (1) with virtual bond dimension D and physical bond dimension d , scales as $\mathcal{O}(NdD^3)$.

d) (3 pts.) Consider a *periodic* MPS

$$|\psi\rangle = \sum_{i_1, \dots, i_N} A_{\alpha_0 \alpha_1}^{i_1} A_{\alpha_1 \alpha_2}^{i_2} A_{\alpha_2 \alpha_3}^{i_3} \dots A_{\alpha_{N-2} \alpha_{N-1}}^{i_{N-1}} A_{\alpha_{N-1} \alpha_0}^{i_N} |i_1, \dots, i_N\rangle, \quad (2)$$

where all the tensors are rank-3 and the first and the last one are contracted with each other. What is the cost of computing $\langle\psi|\psi\rangle$?

3. DMRG for the transverse-field Ising model (10 pts. + bonus)

In this exercise you will use the `Julia` libraries `ITensors` and `ITensorMPS` to study the one-dimensional transverse-field Ising model (TFIM) with the density-matrix renormalization group (DMRG), which variationally minimizes the energy over the MPS of bounded virtual bond dimension D . On a chain of L sites with open boundary conditions the Hamiltonian is

$$\mathcal{H} = - \sum_{n=1}^{L-1} \sigma_n^z \sigma_{n+1}^z - J \sum_{n=1}^L \sigma_n^x, \quad (3)$$

where σ_n^x, σ_n^z are Pauli matrices acting on site n . Here J is the transverse field. For $J < 1$ the system is ferromagnetic, for $J > 1$ it is paramagnetic, and the quantum critical point is $J_c = 1$. DMRG variationally minimizes the energy over matrix-product states of bounded bond dimension D [1]. Use the skeleton in `dmrg_exercise.ipynb`; the helper routines for running DMRG and measuring observables are provided.

- (3 pts.) Complete the `OpSum` representation of (3) and convert it to a Matrix Product Operator (MPO). Validate the implementation at $L = 2$ and $J = 1$: DMRG must reproduce the exact ground-state energy $E_0 = -\sqrt{5}$. Explain where this exact value comes from.
- (4 pts.) Use $L = 40$ and $D = 32$. Run DMRG at $J = 0.5$ and $J = 2$. For each field, report the bulk average of $\langle\sigma_n^x\rangle$ and the long-distance correlator $\langle\sigma_{L/4}^z \sigma_{3L/4}^z\rangle$. Which result is ferromagnetic and which is paramagnetic? Explain how the two observables distinguish the phases. Use the correlator, rather than $\langle\sigma_n^z\rangle$, as the finite-chain diagnostic of ferromagnetic order.
- (3 pts.) At the critical point $J = 1$ and $L = 40$, repeat DMRG for $D = 2, 4, 8, 16$. Tabulate the energy density E_0/L and variance density

$$\frac{\text{var}(\mathcal{H})}{L} = \frac{\langle\mathcal{H}^2\rangle - \langle\mathcal{H}\rangle^2}{L}.$$

Describe their convergence with D . Why is the energy variational, and what does a vanishing variance imply?

- BONUS (0 pts.).** Compute the entanglement-entropy profile $S(\ell)$ at $J = 1$ and at $J = 2$. Contrast the critical profile with the saturation expected in a gapped phase.

- e) **BONUS (0 pts.)**. At $J = 1$, fit the entropy to the open-chain Calabrese–Cardy law [2]

$$S(\ell) = \frac{c}{6} \log \left[\frac{2L}{\pi} \sin \left(\frac{\pi \ell}{L} \right) \right] + \text{const} \quad (4)$$

and compare the fitted central charge with the Ising value $c = 1/2$.

- f) **BONUS (0 pts.)**. At $J = 1$, plot $\langle \sigma_i^z \sigma_{i+r}^z \rangle$ from a site near the centre on log–log axes. Compare the observed decay with the exact Ising prediction $r^{-1/4}$ [3].

References

- [1] S. R. White, “Density matrix formulation for quantum renormalization groups,” Phys. Rev. Lett. 69, 2863 (1992).
- [2] P. Calabrese and J. Cardy, “Entanglement entropy and quantum field theory,” JHEP 06 (2004) 002.
- [3] P. Pfeuty, “The one-dimensional Ising model with a transverse field,” Ann. Phys. 57, 79 (1970).